Modern Cryptography - An Introduction

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Who am I?

- Will Song
  - Competitive math nerd turned CTF player
  - Intern at AIS Denver
  - UIUC SIGPwny
  - 1064CBread

Topics For Today

- RSA
- Attacks on RSA and why you shouldn't roll your own crypto
- Diffie-Hellman
- Attacks on DH and why you shouldn't roll your own crypto
- Elliptic Curves
- Attacks on ECC and why you shouldn't roll your own crypto
- Elliptic Curve Diffie Hellman
- Attacks on ECDH and why you shouldn't roll your own crypto

Prerequisites - Useful Things to Know

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- (Chinese Remainder Theorem) Let p, q be two positive integers such that $\gcd(p, q) = 1$. Then the system of modular equalities

$$x \equiv a \pmod{p}$$
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has exactly one solution modulo pq. For the math savvy people, we say that there is an isomorphism between $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}/pq\mathbb{Z}$.

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- (Bézout) There exist integers x, y such that $ax + by = \gcd(a, b)$.

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- To encrypt a message m, compute $c \equiv m^e \pmod{N}$.
- To decrypt a message c, compute $m \equiv c^d \pmod{N}$.
- Ideally, it should be very hard to find d so we pick two very large primes such that N is approximately 2048 bits or higher.
- Often times people will take $e = 65537 = 2^{16} + 1$ to make encryption easier.

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- Hooray!

RSA - On Your Own

- You need PyCrypto for this.
-
```
from Crypto.PublicKey import RSA
from Crypto.Cipher import PKCS1_OAEP
k = RSA.generate(2048)
print "(%d, %d, %d, %d, %d)" % (k.n, k.e, k.d, k.p, k.q)
print k.encrypt(1337L, 0) # bad, use Crypto.Cipher
c = PKCS1_OAEP.new(k)
print c.encrypt("asdf")
```

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  - (Wiener's Attack) $\dfrac{e}{N}$ has a continued fraction of the form $a_0 + \dfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$. We check convergents $x_n = \dfrac{k_n}{d_n}$ where $x_1 = \dfrac{1}{a_1}, x_2 = \dfrac{1}{a_1 + \frac{1}{a_2}}, \ldots$, and one of the d_n should be our desired d. This works for precisely $d < \dfrac{1}{3} N^{\frac{1}{4}}$.

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  - (Boneh-Durfee) By choosing a specific set of polynomials, we can perform Lenstra-Lenstra-Lovász (LLL) lattice reduction on a polynomial lattice to find a polynomial that contains d as a small root, overall taking polynomial time. This is possible due to a lemma by Hargrave and Graham. This works for precisely $d < N^{0.292}$, which is significantly better than Wiener's approach.

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  If $\gcd(N_i, N_j) \neq 1$, then we can factor either N_i or N_j and easily compute d and thus m, so assume $\gcd(N_i, N_j) = 1$ for all i, j. But this is CRT!!! We can compute $m^e \pmod{\prod_i N_i}$, but $m < N_1$ and $\prod_i N_i > N_1^e > m^e$. This means solving CRT gives you precisely m^e, so we can take the e-th root and get back m.

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- There are many more ways of getting cheesed in RSA.
- Takeaways? Don't roll your own crypto. Use the peer-reviewed library for your language of choice, which will often times be NaCl/libsodium.

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  - Alice and Bob agree on a prime p and some generator g.
  - Alice picks secret exponent a and Bob picks secret exponent b.
  - Alice sends Bob $g^a \pmod{p}$ and Bob sends Alice $g^b \pmod{p}$.
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- You can trivially MITM this, but that is beyond the scope of this talk.

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- Compute $g^{ab} \equiv 76^{69} \equiv 45 \pmod{p}$ and $g^{ba} \equiv 73^{42} \equiv 45 \pmod{p}$.

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- If you're lazy, sometimes it's just not big enough.
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  - (Baby Step Giant Step) We store (j, g^j) for $0 \le j \le \sqrt{p}$. We can check if some function f satisfies $f^k(h) = g^j$ for some $0 \le k \le \sqrt{p}$ and j in the table and if found, we can produce the correct x.

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  - Not usually a problem unless you're too lazy to make/use a multiprecision integer library.

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  - Occasionally the attacker will choose a g such that $g^x \pmod p$ does not have many values, and can brute force the shared secret.
- And other ways of attacking the Discrete Log Problem if you aren't careful. Validate your inputs and use big enough numbers and you shouldn't have any problems.

Prerequisites - Algebra

- A group is a set G equipped with an closed associative binary operation $*$ such that
  - There exists $e \in G$ such that $e * g = g * e = g$ for all $g \in G$.
  - For each $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.
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  - The $p - 1$ non-zero remainders modulo p under multiplication form a group because $\gcd(a, p) = 1$ implies the existence of integers x, y such that $ax + py = 1$, or $ax \equiv 1 \pmod{p}$.
- We don't require the group operation to commute, but when it does we say the group is abelian, or commutative.

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- A ring R is a set equipped with two closed associative binary operations, $+$ and $\cdot$, such that:
  - There exists $0 \in R$ such that $r + 0 = 0 + r = r$ for all $r \in R$.
  - For each $r \in R$, there exists $-r \in R$ such that $r + (-r) = 0$.
  - There exists $1 \in R$ such that $r \cdot 1 = 1 \cdot r = r$ for all $r \in R$.
  - $r_1(r_2 + r_3) = r_1r_2 + r_1r_3$ for all $r_1, r_2, r_3 \in R$.
- A field F is a ring but every non-zero element $f \in F$ has an inverse f^{-1} such that $ff^{-1} = f^{-1}f = 1$.
- Can anyone give any examples of rings and or fields?

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- Can anyone give any examples of rings and or fields?
- We will focus on $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, or the field/ring of numbers modulo p where p is prime.

Elliptic Curves - Definition

- An elliptic curve over the field F, $\mathcal{E}(F)$, is defined as the set of points in $(x, y) \in F^2$ that satisfy
$$y^2 = x^3 + ax + b$$
for some $a, b \in F$.
In particular, we also want this thing to be non-singular, or some value called the discriminant (very similar to the discriminant of a quadratic equation) to be non-zero.

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- For crypto purposes, we will use $F = \mathbb{F}_q$ where $q = p^k$ for some prime p, but often times we will see $q = p$. In fact, I haven't introduced what $\mathbb{F}_q$ for $k > 1$ even looks like so we won't be seeing it yet.

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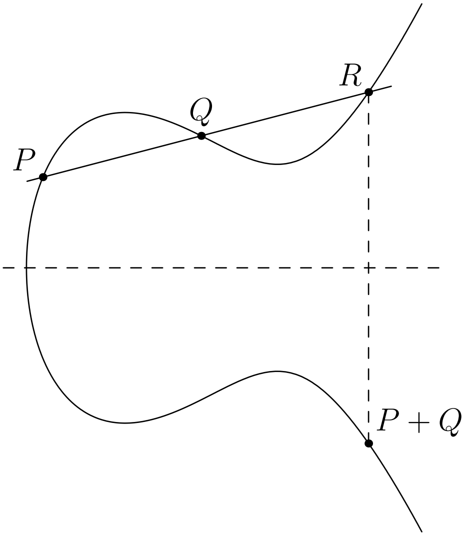
- An elliptic curve over the field F, $\mathcal{E}(F)$, is defined as the set of points in $(x, y) \in F^2$ that satisfy
$$y^2 = x^3 + ax + b$$
for some $a, b \in F$.
In particular, we also want this thing to be non-singular, or some value called the discriminant (very similar to the discriminant of a quadratic equation) to be non-zero.
- For crypto purposes, we will use $F = \mathbb{F}_q$ where $q = p^k$ for some prime p, but often times we will see $q = p$. In fact, I haven't introduced what $\mathbb{F}_q$ for $k > 1$ even looks like so we won't be seeing it yet.
- We want to turn this thing into a group. To do that, we need a picture.
- If you want to sound smart, you can also call this type of object an abelian variety (it is a specific type of abelian variety), or a non-singular projective cubic.

Elliptic Curves in $\mathbb{R}$ - The Picture



Special Cases

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  - Taking the tangent line to P seems intuitive enough.
- With respect to the previous image, $P + Q + R$ doesn't seem to exist!
  - We switch into $\mathbb{RP}^2$.
  - Define $0 = \infty$.

Elliptic Curves in $\mathbb{R}$ - The Formulas

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- The line definition works every time because algebra.
  - Let $P = (P_x, P_y), Q = (Q_x, Q_y)$.
  - We can compute $\lambda = \frac{P_y - Q_y}{P_x - Q_x}$ if $P \neq Q$ and $\lambda = \frac{3x^2 + a}{2y}$ otherwise, as well as

$$
\begin{aligned}
(P+Q)_x &= \lambda^2 - (P_x + Q_x) \\
-(P+Q)_y &= \lambda((P+Q)_x - P_x) + P_y.
\end{aligned}
$$

- Computing multiples $n \cdot P$ is easy with a method similar to exponentiation by squaring.

Elliptic Curves - The Formulas

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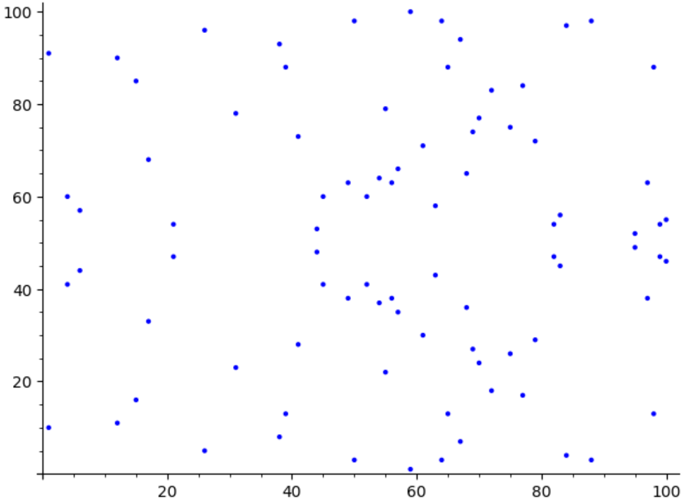
Elliptic Curves - The Formulas

- But how do we do this in $\mathbb{F}_p$?
- Notice how every non-zero element of $\mathbb{F}_p$ has an inverse? Yeah dividing is multiplying by the inverse (yay high school).
- You can't get a very good picture of this in $\mathbb{F}_p$ which is why we tend to draw it in $\mathbb{R}$ and then just copy over the intuition.
- The math of elliptic curves is hard and you surely don't want to do it yourself. Use a library for it.

Elliptic Curves - On Your Own

- You will need either SageMath or some fraction of packages that come with Sage.
- ```
  E = EllipticCurve(GF(101), [1, -3]) # EllipticCurve(F, [A, B])
  print E.is_ordinary()
  print E.gens()[0].xy()
  E.plot()
  ```
  ```

Elliptic Curves - On Your Own



Elliptic Curves - Pitfalls

- Remember the DUAL_EC_DRBG scandal? Here's how the backdoor worked.
  - Start with your curve $\mathcal{E}(\mathbb{F}_p)$ and two public points P, Q. Let $\pi_x : \mathcal{E}(\mathbb{F}_p) \to \mathbb{F}_p$ be the projection onto the x-axis. Let $T : \mathbb{F}_p \to \mathbb{F}_p$ truncate an integer to 240 bits (we are working with a 256 bit prime). Let $f_1(s) = sP$, $f_2(s) = sQ$. Let s_0 be the initial seed. We will produce seeds $s_1, s_2, \ldots$ and random numbers $r_1, r_2, \ldots$ like so.

$$
\begin{tikzcd}
s_0 \arrow[r, "\pi_x \circ f_1"] & s_1 \arrow[r, "\pi_x \circ f_1"] \arrow[d, "T \circ \pi_x \circ f_2"] & s_2 \arrow[r, "\pi_x \circ f_1"] \arrow[d, "T \circ \pi_x \circ f_2"] & \cdots \arrow[d, "T \circ \pi_x \circ f_2"] \\
 & r_1 & r_2 & \cdots
\end{tikzcd}
$$

Elliptic Curves - Pitfalls

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    - Suppose you retrieve $R = f_2(s_1) = s_1 Q$. What can you do?

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  - We can do this in 2^{16} brute force attempts (notice how the sign of y makes no difference to the resulting x).
- Lesson? Don't trust the NSA to not spy on you.

Elliptic Curves - Extra

- Elliptic Curves offer similar security levels to RSA at significantly smaller key sizes and parameters.
- Around a year ago there was a seemingly inane math question posted to several Facebook groups that boiled down to the following.
  Find integers x, y, z satisfying

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 4.$$

  I haven't taught you why this is equivalent to an elliptic curve but if you're interested you can play around with it and try to find the connection. (Hint: the smallest x, y, z are roughly 80 digits long)

Elliptic Curve Diffie-Hellman

- Earlier we noted that we used a group for regular Diffie-Hellman. We remark that we can do the same with elliptic curves, because they form a group!
  - Alice and Bob agree on a field $\mathbb{F}_p$, a curve $\mathcal{E}(\mathbb{F}_p)$, and a point P.
  - Alice picks a random number a and Bob picks a random number b.
  - Alice sends Bob aP and Bob sends Alice bP.
  - Both parties compute $K = (ab)P = (ba)P$.

Elliptic Curve Diffie-Hellman - Pitfalls

- A lot of the pitfalls of regular DH carry over. If the number of points on the curve is too small, BSGS can solve it relatively quickly. If the number of points is smooth, we can use Pohlig-Hellman.

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- We also have some new issues.
- If the curve in question has a prime number of points, there is an efficient algorithm to solve the ECDLP.
  - (Smart) We lift $\mathcal{E}(\mathbb{F}_p)$ to $\mathcal{E}(\mathbb{Q}_p)$ over the p-adic numbers via the natural embedding and then perform a Hensel Lift via Hensel's Lemma to lift the two curve points in question to $\mathcal{E}(\mathbb{Q}_p)$. Some math in $\mathbb{Q}_p$ gives us the desired exponent.

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  - Let $t = p + 1 - \#\mathcal{E}(\mathbb{F}_p)$. We call this the trace of Frobenius of the curve $\mathcal{E}(\mathbb{F}_p)$. Given a desired trace of Frobenius t, we are able to produce curves with $p + 1 \pm t$ points via the theory of Complex Multiplication and class field theory, which is also more math, so use a library.

Conclusion

- tl;dr cryptography is hard. Please don't try to write your own library. You will screw up some way or another.

Resources

- picoCTF
- plaidCTF
- uiuctf
- CSAW CTF
- My Github (ctf solutions/source)
- Boneh-Durfee and Coppersmith
- SageMath
- Rational Points on Elliptic Curves (Silverman & Tate)
- Abstract Algebra (Dummit & Foote)

Q & A

- Any questions? Any enthusiasm for wanting to get murdered learning the math behind my Underhanded Crypto Challenge submission?