CS/ECE 374 A ✧ Spring 2018

Homework 10

Due Tuesday, April 24, 2018 at 8pm

This is the last graded homework before the final exam.

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1. (a) A subset S of vertices in an undirected graph G is ***half-independent*** if each vertex in S is adjacent to *at most one* other vertex in S. Prove that finding the size of the largest half-independent set of vertices in a given undirected graph is NP-hard.

   (b) A subset S of vertices in an undirected graph G is ***sort-of-independent*** if if each vertex in S is adjacent to *at most 374* other vertices in S. Prove that finding the size of the largest sort-of-independent set of vertices in a given undirected graph is NP-hard.

2. Fix an alphabet $\Sigma = \{0, 1\}$. Prove that the following problems are NP-hard.[1]

   (a) Given a regular expression R over the alphabet Σ, is $L(R) \neq \Sigma^*$?

   (b) Given an NFA M over the alphabet Σ, is $L(M) \neq \Sigma^*$?

   *[Hint: Encode all the **bad** choices for some problem into a regular expression R, so that if **all** choices are bad, then $L(R) = \Sigma^*$.]*

3. Let $\langle M \rangle$ denote the encoding of a Turing machine M (or if you prefer, the Python source code for the executable code M). Recall that $x \bullet y$ denotes the concatenation of strings x and y. Prove that the following language is undecidable.

$$\text{SelfSelfAccept} := \{\langle M \rangle \mid M \text{ accepts the string } \langle M \rangle \bullet \langle M \rangle\}$$

   Note that Rice's theorem does *not* apply to this language.

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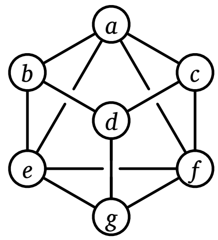
[1] In fact, both of these problems are NP-hard even when $|\Sigma| = 1$, but proving that is much more difficult.

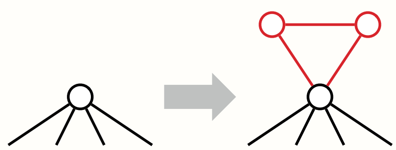
Solved Problem

4. A *double-Hamiltonian tour* in an undirected graph G is a closed walk that visits every vertex in G exactly twice. Prove that it is NP-hard to decide whether a given graph G has a double-Hamiltonian tour.



This graph contains the double-Hamiltonian tour $a\to b\to d\to g\to e\to b\to d\to c\to f\to a\to c\to f\to g\to e\to a$.

Solution: We prove the problem is NP-hard with a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a small gadget to every vertex of G. Specifically, for each vertex v, we add two vertices $v^\sharp$ and $v^\flat$, along with three edges $vv^\flat$, $vv^\sharp$, and $v^\flat v^\sharp$.



A vertex in G, and the corresponding vertex gadget in H.

I claim that G has a Hamiltonian cycle if and only if H has a double-Hamiltonian tour.

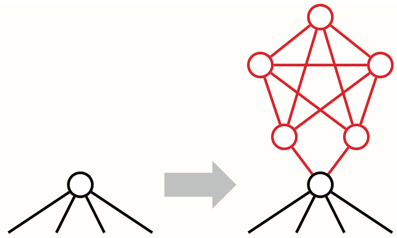
$\Longrightarrow$ Suppose G has a Hamiltonian cycle $v_1\to v_2\to\cdots\to v_n\to v_1$. We can construct a double-Hamiltonian tour of H by replacing each vertex v_i with the following walk:

$$\cdots\to v_i\to v_i^\flat\to v_i^\sharp\to v_i^\flat\to v_i^\sharp\to v_i\to\cdots$$

$\Longleftarrow$ Conversely, suppose H has a double-Hamiltonian tour D. Consider any vertex v in the original graph G; the tour D must visit v exactly twice. Those two visits split D into two closed walks, each of which visits v exactly once. Any walk from $v^\flat$ or $v^\sharp$ to any other vertex in H must pass through v. Thus, one of the two closed walks visits only the vertices v, $v^\flat$, and $v^\sharp$. Thus, if we simply remove the vertices in $H \setminus G$ from D, we obtain a closed walk in G that visits every vertex in G once.

Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

With more effort, we can construct a graph H that contains a double-Hamiltonian tour ***that traverses each edge of H at most once*** if and only if G contains a Hamiltonian cycle. For each vertex v in G we attach a more complex gadget containing five vertices and eleven edges, as shown on the next page.



A vertex in G, and the corresponding modified vertex gadget in H.

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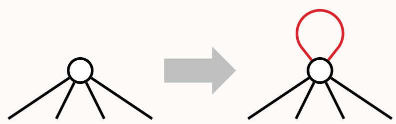
Non-solution (self-loops): We attempt to prove the problem is NP-hard with a reduction from the Hamiltonian cycle problem. Let G be an arbitrary undirected graph. We construct a new graph H by attaching a self-loop every vertex of G. Given any graph G, we can clearly construct the corresponding graph H in polynomial time.

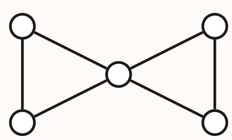


An incorrect vertex gadget.

Suppose G has a Hamiltonian cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$. We can construct a double-Hamiltonian tour of H by alternating between edges of the Hamiltonian cycle and self-loops:

$$v_1 \to v_1 \to v_2 \to v_2 \to v_3 \to \cdots \to v_n \to v_n \to v_1.$$

On the other hand, if H has a double-Hamiltonian tour, we *cannot* conclude that G has a Hamiltonian cycle, because we cannot guarantee that a double-Hamiltonian tour in H uses *any* self-loops. The graph G shown below is a counterexample; it has a double-Hamiltonian tour (even before adding self-loops!) but no Hamiltonian cycle.



This graph has a double-Hamiltonian tour.

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Rubric: 10 points, standard polynomial-time reduction rubric